A.3 Kohn, virial and Bertrand theorems

In this annex, we present derivations of three theorems used in the main text, namely the Kohn, virial and Bertrand's theorems. The demonstration of the latter is an original work, inspired by several proofs reported in the literature.

A.3.1 Kohn theorem

Kohn's theorem is a very strong result to study particles trapped in a harmonic potential and was first formulated to describe the cyclotron motion of particles in a static, uniform magnetic field [Kohn 1961]. It states that regardless of the inter-particles interactions, the center of mass of the distribution oscillates at the cyclotron frequencies. Its extension in [Brey *et al.* 1989] generalized this result to the case of identical particles in a harmonic trap. Later, in [Dobson 1994], the "harmonic potential theorem" demonstrates the rigid transport of the many-body wavefunction.

Proof

We describe trapped particles with the Hamiltonian

$$H = H_{\text{kin}} + H_{\text{trap}} + H_{\text{int}} = \sum_i \frac{p_i^2}{2m} + \sum_i \frac{1}{2} m\omega^2 r_i^2 + \sum_{i,j} w\left(\mathbf{r}_i - \mathbf{r}_j\right). \tag{A.26}$$

We introduce the center of mass variables $\mathbf{R} = \sum_i \mathbf{r}_i / N$ and $\mathbf{P} = \sum_i \mathbf{p}_i$, as well as the relative coordinates $\boldsymbol{\xi}_i = \mathbf{r}_i - \mathbf{R}$ and $\mathbf{q}_i = \mathbf{p}_i - \mathbf{P}/N$. With these notations, we can rewrite the Hamiltonian (A.26) as:

$$H = H_{\text{CoM}} + H_{\text{rel}}, \tag{A.27a}$$

$$H_{\text{CoM}} = \frac{P^2}{2Nm} + \frac{1}{2} N m\omega^2 R^2, \tag{A.27b}$$

$$H_{\text{rel}} = \sum_i \frac{q_i^2}{2m} + \sum_i \frac{1}{2} m\omega^2 \xi_i^2 + \sum_{i,j} w\left(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j\right). \tag{A.27c}$$

Note that relative- and center of mass- coordinates commute and $[\mathbf{R}, \mathbf{P}] = i\hbar$. We have therefore $[H_{\text{rel}}, \mathbf{R}] = [H_{\text{rel}}, \mathbf{P}] = 0$, and the center of mass follows the simple Hamiltonian of a trapped particle with mass Nm in a harmonic potential.

The same proof can be applied for particles with a charge q in an additional static or uniform magnetic field $\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{A}$ and uniform electric field $\mathbf{E}(t)$. The center of mass Hamiltonian is then given by

$$H_{\text{CoM}} = \frac{(\mathbf{P} - Q\mathbf{A})^2}{2M} + \frac{1}{2} M\omega^2 R^2 + Q\mathbf{E}(t).\mathbf{R}, \tag{A.28}$$

with $Q = Nq$ and $M = Nm$.

The theorem can be further extended to show that the wave function of the N particles system can be expressed as the product of a center of mass wave function and an "internal" wave function. To do so, we must introduce the Jacobi coordinates $\mathbf{u}_1 = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{u}_2 = \frac{\mathbf{r}_1+\mathbf{r}_2}{2} - \mathbf{r}_3$, ..., $\mathbf{u}_n = \frac{\mathbf{r}_1+...+\mathbf{r}_n}{n} - \mathbf{r}_{n+1}$ and the corresponding momenta. Those coordinates do verify the canonical commutation relations and we can show that they can be used to rewrite H_{rel}, hence the result.

A.3.2 Virial theorem

Virial theorem was formulated by Rudolf Clausis in 1870 [Clausius 1870] and applies to particles in a potential $V(\mathbf{r})$ such that

$$V(\lambda \mathbf{r}) = \lambda^n V(\mathbf{r}). \tag{A.29}$$

In particular, this class includes all potential with the form $V(\mathbf{r}) = \left(\alpha_x x^p + \alpha_y y^p + \alpha_z z^p\right)^q$, with $n = pq$.

The theorem can be formulated as follows: for any ensemble submitted to a potential (A.29), the kinetic energy E_{K} and the potential energy E_P in the steady states are related through the relation:

$$2E_{\text{K}} = nE_{\text{P}} \tag{A.30}$$

Virial theorem is often used in stellar physics and provides for instance a derivation of Chandrasekhar limit for the stability of white dwarf stars.

Lemma *Euler Theorem*

For any potential, if $\epsilon \ll 1$, we can write $V((1+\epsilon)\mathbf{r}) \simeq V(\mathbf{r}) + \epsilon\,(\mathbf{r}.\partial_{\mathbf{r}})\,V(\mathbf{r})$

Since $V((1+\epsilon)\mathbf{r}) = (1+\epsilon)^n V(\mathbf{r}) \simeq V(\mathbf{r}) + n\epsilon V(\mathbf{r})$, this leads to Euler theorm for homogeneous functions:

$$(\mathbf{r}.\partial_{\mathbf{r}})\,V(\mathbf{r}) = nV(\mathbf{r}) \tag{A.31}$$

Proof We describe an ensemble of particles in a potential V with the distribution function $f(\mathbf{r}, \mathbf{p}, t)$. The evolution of f is governed by Liouville equation

$$\partial f = \mathcal{L} f,$$

with the Liouville operator being $\mathcal{L} = \frac{\mathbf{p}}{m}.\partial_{\mathbf{r}} - (\partial_{\mathbf{r}} V)\,.\partial_{\mathbf{p}}$.

Let us consider the quantity $C(t) = \langle \mathbf{r}.\mathbf{p} \rangle = \int f\,\mathbf{r}.\mathbf{p}$. In steady state, $\partial_t C = 0$ and we can express

$$0 = \int \partial_t f\;\mathbf{r}.\mathbf{p} = \int \mathcal{L}\,(f)\;\mathbf{r}.\mathbf{p} = -\int f\,\mathcal{L}\,(\mathbf{r}.\mathbf{p}) \tag{A.32}$$

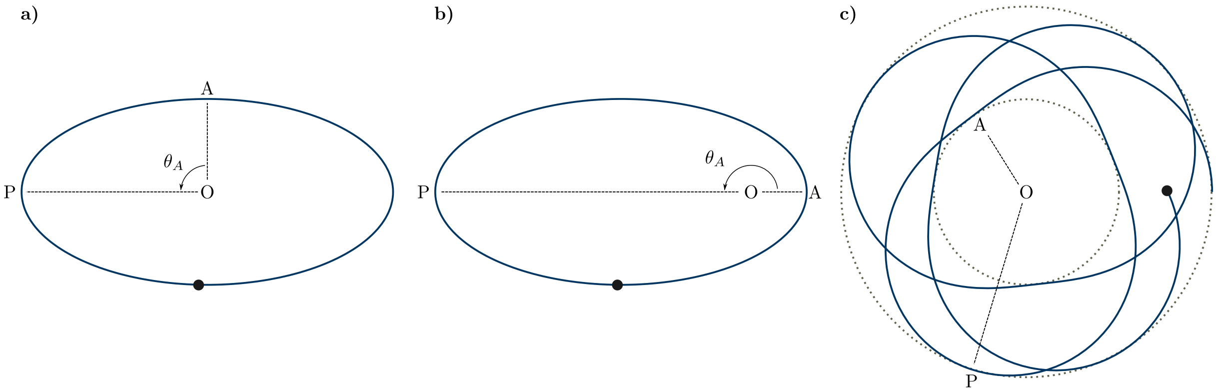

Figure A.3: Three orbits in central potentials. In a), the Hook potential results in an apside angle $\theta_A = \pi/2$. In b), the Coulomb potential gives $\theta_A = \pi$. In c), an arbitrary potential gives a rosette-like trajectory.

where we integrated by parts. Using Euler theorem, we can express $\mathcal{L}(\mathbf{r}.\mathbf{p}) = p^2/m - nV$, and the previous condition leads to

$$\int f \frac{p^2}{m} = n \int f V \tag{A.33}$$

hence the virial theorem.

A.3.3 Bertrand's theorem

Bertrand's theorem describes the behaviour of particles trapped in a central force potential $V(r)$. The theorem states that, even if any arbitrary potential admits a circular orbit, only two are such that all bound orbits are closed: the Coulomb potential $V_C(\mathbf{r}) = -\kappa/r$ and the Hook potential $V_H(\mathbf{r}) = \kappa' r^2$.

The corollary of this theorem is that any potential apart from V_C and V_C will admit rosette-like orbits, as the particle will cover densely all positions between the aphelion and the perihelion (see Fig. A.3)

The theorem was first formulated in 1873 by J. Bertrand [Bertrand 1873] (see [Santos *et al.* 2011] for an english translation) as a general analysis of bound orbits. It was proposed as a tool for the study of the three-body problem and can be used as an alternative demonstration to the shape of Newton's gravitational potential, since all trajectories observed among celestial bodies are closed.

Several proofs to Bertrand's theorem can be found nowadays. Most of them have the same structure and only differ in the last step of the demonstration. The original proof uses a global approach; a perturbative method can also be applied [Goldstein *et al.* 2001] or additionnal constant of motion can be invoked [Martinez-y Romero *et al.* 1992]. Many more rely on an inverse transform [Grandati *et al.* 2008, Santos *et al.* 2009]. Here, we show a global approach inspired by [Arnold *et al.* 1997].

Few words of vocabulary, illustrated on figure A.3:

- The apside angle θ_A is the angle $\widehat{AOP}$ between the the aphelion A, the center O et the perihelion P.
- An orbite is closed if the trajectory followed by a particle will eventually loop. Equivalently, this condition corresponds to the apside angle being commensurate with π: $\theta_A = \frac{m}{n}\pi$.

Step 1 *If all orbits are closed, than the apside angle must be the same for all trajectories*

Let us consider a closed orbit. As mentioned above, this condition requires that the apside angle of this trajectory is commensurate with π.

If the apside angle was not the same for all orbits, but depended continuously of the energy or angular momentum of the trajectory, it would necessarily take values incommensurate with π according to the intermediate value theorem.

Therefore, if all orbits are closed, they must all have the same apside angle θ_A.

Step 2 *For almost circular orbits, the only central potentials with constant apside angles are* $V(r) = \kappa \ln r/r_0$ *and* $V(r) = \kappa r^\alpha$ *with* $\alpha > -2$ *and* $\alpha \neq 0$. *The corresponding apside angles are* $\theta_A = \frac{\pi}{\sqrt{2+\alpha}}$

We consider a point-like particle with mass m subject to the force $\mathbf{F}(r) = -\partial_r V(r)\mathbf{u}_r$ and decribe its trajectory in polar coordinates (r, θ).

The energy conservation can be expressed as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r). \tag{A.34}$$

Using the conservation of angular momentum $\mathbf{L} = mr^2\dot{\theta}\mathbf{u}_z$, we can express the kinetic energy as a function of the distance r and of its angular dependance:

$$\dot{r}^2 + r^2\dot{\theta}^2 = \left(\frac{L_z}{mr^2}\frac{dr}{d\theta}\right)^2 + \frac{L_z^2}{2mr^2}. \tag{A.35}$$

We follow Binet's notation $u = r^{-1}$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$ and we can write eq.(A.34) as

$$E = \frac{L_z^2}{2m}\left(\frac{du}{d\theta}\right)^2 + V_{\text{eff}}(u), \tag{A.36}$$

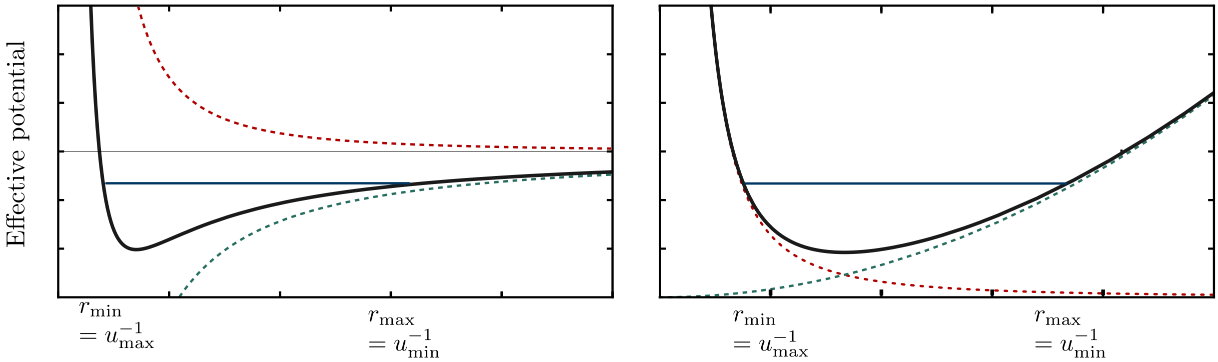

Figure A.4: Effective potentials (black line), expressed as the sum of the kinetic (red dashed line) and potential (green dashed line) energies, for $\alpha < 0$ (left) and $\alpha > 0$ (right). For a given energy (blue horizontal line), the aphelion ($r_{\min}$) and perihelion ($r_{\max}$) define the accessible position range.

where we introduced (see Fig. A.4)

$$V_{\text{eff}}(u) = \frac{L_z^2}{2m}u^2 + V(u^{-1}). \tag{A.37}$$

Let us consider a circular trajectory u_0 with energy E_0 and momentum L_z and a stable orbit close to it, such that $u = u_0 + \rho(\theta)$. Up to the second order, $V_{\text{eff}}(u) = V_{\text{eff}}(u_0) + \frac{1}{2}\rho^2 V''_{\text{eff}}(u_0)$ since $V'_{\text{eff}}(u_0) = 0$ and eq. (A.36) reads

$$E = \frac{L_z^2}{2m}\left(\frac{d\rho}{d\theta}\right)^2 + \frac{1}{2}\rho^2 V''_{\text{eff}}(u_0), \tag{A.38}$$

which describes an oscillation with the angular frequency

$$\Omega = \sqrt{\frac{mV''_{\text{eff}}(u_0)}{L_z^2}} = \sqrt{\frac{3V'(r_0) + r_0 V''(r_0)}{V'(r_0)}} > 0, \tag{A.39}$$

where we used $V'_{\text{eff}}(u_0) = 0 \Rightarrow V'(r_0) = L_z^2/mr_0^3$ and expressed the stability of the orbit.

Without any loss of generality, we can chose the origin such that $u = u_0 + A\cos\Omega\theta$. The aphelion is reached for $\theta = 0$ and the perihelion for $\theta = \theta_A = \pi/\Omega$. The condition expressed in step 1 thus leads to

$$\frac{3V'(r) + rV''(r)}{V'(r)} = 2 + \alpha, . \tag{A.40}$$

with $\alpha > -2$. For $\alpha = 0$, eq. (A.40) leads to $V(r) = \kappa \ln r/r_0$ and $\theta_A = \pi/\sqrt{2}$, and we can rule out this potential since its aspide angle is incommensurate with π. For $\alpha \neq 0$, we find $V(r) = \kappa' r^\alpha$ and

$$\theta_A = \frac{\pi}{\sqrt{2 + \alpha}}. \tag{A.41}$$

Step 3 *The only possible values for the apside angle to be commensurate with π are $\alpha = 2$ and $\alpha = -1$, hence Bertrand's theorem*

For each value of α, since the apside angle is the same for all trajectories, we will restrict the study to a convenient situation and express the apside angle using eq. (A.36)

$$\theta_A \quad = \quad \frac{L_z u_{\max}}{\sqrt{2m}} \int_{\frac{u_{\min}}{u_{\max}}}^{1} \frac{dx}{\sqrt{E - V_{\text{eff}}(x u_{\max})}}. \tag{A.42}$$

For $\alpha > 0$, we consider an orbit with an energy $E \to +\infty$. The aphelion is approaching zero as $u_{\max} = r_{\min}^{-1} \to \infty$, while the perihelion as located at $u_{\min} = 0$ (see Fig.A.4). We can thus consider $V\left(1/u_{\max}\right) \to 0$ and therefore $E \simeq \frac{L_z^2}{2m} u_{\max}^2$. We can express the effective potential as:

$$V_{\text{eff}}(x u_{\max}) = E x^2 + \frac{L_z^\alpha}{(2m)^{\alpha/2} E^{\alpha/2}} \frac{\kappa}{x^\alpha} \underset{E\to\infty}{\simeq} E x^2 \tag{A.43}$$

Considering $\frac{u_{\min}}{u_{\max}} \to 0$, eq. (A.42) leads to

$$\theta_A \underset{E\to\infty}{\longrightarrow} \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \pi/2. \tag{A.44}$$

Comparing eq. (A.44) and (A.41), the only possible potential is the Hook potential, corresponding to a harmonic motion with $\alpha = 2$.

For $\alpha < 0$, orbits can exist only if $\kappa < 0$ and $E < 0$. With such prerequisites, we consider an orbit with an energy $E \to 0^-$. The aphelion is then given by $-\kappa u_{\max}^{-\alpha} = \frac{L_z^2}{2m} u_{\max}^2$ while the perihelion as located at $u_{\min} = 0$ (see Fig. A.4). The apside angle is thus converging towards

$$\theta_A \underset{E\to 0}{\longrightarrow} \frac{L_z u_{\max}}{\sqrt{2m}} \int_0^1 \frac{dx}{\sqrt{-\frac{L_z^2}{2m} u_{\max}^2 x^2 - \kappa x^{-\alpha} u_{\max}^{-\alpha}}} = \frac{\pi}{2 + \alpha}. \tag{A.45}$$

Comparing eq. (A.45) and (A.41), the only possible potential is the Coulomb potential, corresponding to an elliptique motion with $\alpha = -1$.