

the Kohn, virial and Bertrand's theorem, with an original approach. Annex A.4 summarizes elements of the elastic collisions theory required to address scattering problems. Eventually, annex A.5 provides the technical references of the devices used on the apparatus.

A.1 Alkaline atoms in magnetic fields

The Zeeman effect accounts for the influence of an external magnetic field on the eigenelements of an atom. The atom is coupled to the field through its magnetic momentum \mathcal{M} , which can be related to the angular momentums (orbital \mathbf{L} , intrinsic \mathbf{S} and nuclear \mathbf{I}) through Wigner-Eckart theorem. As a consequence, a small field will lift the degeneracy of the so called *Zeeman sublevels* which differ only by their projected angular momentum. At a highest order, the impact on the hyperfine splitting must be taken into account. In asymptotic regimes, convenient formulae can be derived to estimate the Zeeman shifts. If the magnetic field is small enough, such that its influence is negligible compared to the hyperfine splitting, the energy shift of each Zeeman sublevel is linear with the field and proportional to its projected momentum. If the magnetic field is strong enough, such that the hyperfine splitting can be neglected, the energy shift is still proportional to the field but only depends of the electron intrinsic orientation. The calculation can be explicitly carried out for $\mathbf{L} = 0$; the resulting Breit-Rabi formula is therefore appropriate for the ground states of alkali atoms and plays an important role in our case.

This appendix reviews and summarizes the main results describing the behavior of alkali atoms such as Lithium and Potassium in a magnetic field.

A.1.1 Wigner-Eckart theorem and Lande factor

The magnetic momentum of an atom is a vectorial operator and follows as such the Wigner-Eckart theorem: The restriction of any vectorial operator \mathbf{V} to an eigenspace of the angular momentum \mathbf{F} is proportional to the angular momentum with a coefficient

$$\frac{\langle \mathbf{V} \cdot \mathbf{F} \rangle_F}{F(F+1)\hbar^2}. \quad (\text{A.1})$$

Here, a vectorial operator is defined by its commutation relations $[F_x, V_x] = 0$, $[F_x, V_y] = i\hbar V_z$ and $[F_x, V_z] = -i\hbar V_y$ (and circular permutations) while the mean value can be estimated on any state of the eigenspace.

Three angular momentum are required to describe an atomic state and the magnetic momentum can therefore be expressed as the sum of three contribution. The proportionality factor between a magnetic momentum and the related angular momentum is expressed with Bohr magneton $\mu_B = \frac{e\hbar}{2m_e} \simeq 1.4 \times h \text{ MHz/G}$ and so called *Landé factor*, usually noted g .

- The orbital momentum \mathbf{L} of the electron is associated to a magnetic momentum $\mathcal{M}_L = -\frac{\mu_B}{\hbar} g_L \mathbf{L}$ with $g_L = 1$.

δ_L	s	p	d	f	g_J	$S_{1/2}$	$P_{1/2}$	$P_{3/2}$
Li	0.40	0.04	0	0	Li	2	0.666	1.335
K	2.19	3.59	2.46	0.02	K	2	2/3	4/3

Table A.1: Landé factors (left) and quantum defects for Lithium and Potassium, as given by [Burkhardt and Leventhal 2010]

^{40}K	a_{hf}	b_{hf}	^6Li	a_{hf}	b_{hf}
$4^2S_{1/2}$	-285.730	-	$2^2S_{1/2}$	152.136	-
$4^2P_{1/2}$	-34.523	-	$2^2P_{1/2}$	17.386	-
$4^2P_{3/2}$	-7.585	-3.445	$2^2P_{3/2}$	-1.155	-0.10

Table A.2: Hyperfine structure of ^6Li and ^{40}K , as given by [Arimondo *et al.* 1977]

- The intrinsic momentum \mathbf{S} of the electron (or spin) is associated to a magnetic momentum $\mathcal{M}_S = -\frac{\mu_B}{\hbar} g_S \mathbf{S}$ with $g_S \simeq 2$. This value is predicted by Dirac equation; magnetic anomalies modify it by $\sim +0.02$.
- The nuclear momentum \mathbf{I} is associated to a magnetic momentum $\mathcal{M}_I = -\frac{\mu_B}{\hbar} g_I \mathbf{I}$. The factor g_I depends of the nuclei but is always much smaller than 1 (1.7×10^{-4} for ^{40}K and -4.5×10^{-4} for ^6Li).

It is often useful to consider composed angular momentum, such as $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and $\mathbf{F} = \mathbf{J} + \mathbf{I}$. We can therefore introduce the corresponding Landé factors

$$g_J = \frac{g_S}{2} \frac{J(J+1) + S(S+1) - L(L+1)}{J(J+1)} + \frac{g_L}{2} \frac{J(J+1) + L(L+1) - S(S+1)}{J(J+1)}, \quad (\text{A.2})$$

$$g_F = \frac{g_J}{2} \frac{F(F+1) + J(J+1) - I(I+1)}{F(F+1)} + \frac{g_I}{2} \frac{F(F+1) + I(I+1) - J(J+1)}{F(F+1)}. \quad (\text{A.3})$$

The Wigner-Eckhart theorem is extremely useful as long as the vectorial operator can be correctly approximated by its restriction to a subspace of the angular momentum. This is correct as long as the variations due to the operator are small compared to the difference between two subspaces. Since S and I are intrinsic quantities, the previous results are always correct. For L (resp. J , F), they hold until the energy shift becomes comparable to the atomic structure of ~ 100 THz (resp fine structure ~ 100 GHz, hyperfine structure ~ 100 MHz)

A.1.2 Zeeman hamiltonian

To account for the Zeeman effect on alkali, we describe an atom with the following Hamiltonian [Cohen-Tannoudji *et al.* 1997],

$$H = H_0 + W_{\text{so}} + W_{\text{hf}} + H_Z, \quad (\text{A.4})$$

where

- H_0 describes a spinless valence electron in the electrostatic potential of the nuclei and core electron cloud. For alkali atoms, the energy of an orbit depends not only on the principal quantum n , but also on the angular momentum \mathbf{L} , as the orbit can bring the valence electron to cross inner electronic layers. This effect can be simply taken into account by the so-called *quantum defect* [Hanle 1984]. The eigenenergies then take the form $E_0(n, L) = -R_y / (n - \delta_{n,L})^2$, where R_y is the Rydberg constant and $\delta_{n,L}$ is the quantum defect, mostly depending on L (see table A.1).
- W_{so} describes the coupling between the spin \mathbf{S} and the orbital angular momentum \mathbf{L} . It can be written, as any interaction lifting the degeneracy of the $\mathbf{J} = \mathbf{S} + \mathbf{L}$ levels, under the form:

$$W_{so} \propto \mathbf{L} \cdot \mathbf{S} \simeq a_f (J(J+1) - L(L+1) - S(S+1)), \quad (\text{A.5})$$

where a_f gives the fine structure splitting.

- W_{hf} accounts for the coupling to the angular momentum of the nuclei, generating the hyperfine structure of the atom:

$$W_{hf} = \frac{a_{hf}}{\hbar^2} \mathbf{I} \cdot \mathbf{J} + \frac{b_{hf}}{\hbar^2} \frac{3(\mathbf{I} \cdot \mathbf{J})^2 + \frac{3}{2} \hbar(\mathbf{I} \cdot \mathbf{J}) - I^2 J^2}{2I(2I-1)J(2J-1)} \quad (\text{A.6a})$$

$$= \frac{a_{hf}}{2} (F(F+1) - I(I+1) - J(J+1)) \text{ if } b_{hf} = 0 \quad (\text{A.6b})$$

where a_{hf} and b_{hf} are the magnetic dipole and electric quadrupole constants respectively.

- H_Z describes the coupling of magnetic momenta to an external magnetic field:

$$\hat{H}_Z = - \left(\hat{\mathcal{M}}_S + \hat{\mathcal{M}}_L + \hat{\mathcal{M}}_I \right) \cdot \mathbf{B} \quad (\text{A.7})$$

A.1.3 Asymptotic behaviors

- At low field: linear Zeeman effect treatment

If the Zeeman shift is much smaller than the hyperfine splitting, the Zeeman effect can be taken into account perturbatively. Atomic states are correctly described by their projected angular momentum m_F and their (degenerate) energies are given by:

$$E(m_F) = E_0(n, L) + a_{hf} (F(F+1) - I(I+1) - J(J+1)) / 2 \quad (\text{A.8})$$

Up to first order in perturbation, the Zeeman effect is given by

$$\Delta E = \langle m_F | H_Z | m_F \rangle = \frac{\mu_B}{\hbar} g_F B m_F \quad (\text{A.9})$$

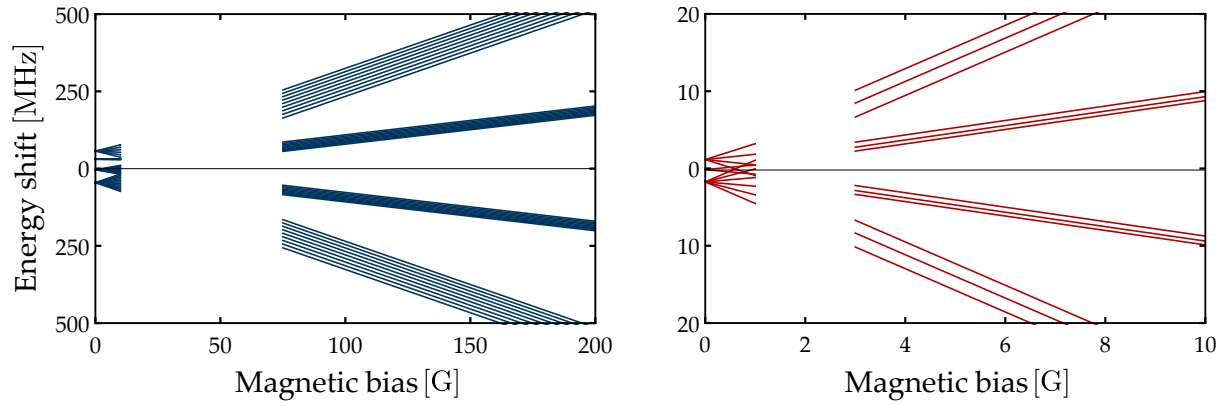


Figure A.1: Asymptotic regimes for the Zeeman shift of the $P_{3/2}$ states of ^{40}K (left) and ^6Li (right) as a function of the magnetic field. The low field regime is given by equation (A.9), where the coupling to the magnetic field is treated perturbatively. The high field regime is given by equation (A.11b), where the hyperfine structure is estimated as a perturbation.

For Lithium as well as Potassium, this regime is only valid as long as the magnetic field does not exceed few Gauss (see Fig. A.1).

- At high field: Paschen-Back regime

If the Zeeman shift is much larger than the hyperfine splitting, the restriction of the Hamiltonian to an eigenspace of the angular momentum \mathbf{F} is not relevant anymore and the projected momentum m_F is not a good quantum number to describe the situation. The magnetic momentum \mathcal{M} cannot be expressed with \mathbf{F} but must be described as the sum of \mathcal{M}_I and \mathcal{M}_J , making m_I and m_J the relevant quantum numbers.

We therefore consider the eigenstates $|n, L, S, J, I, m_J, m_I\rangle$ of the Hamiltonian $H = H_0 + W_{\text{so}} + H_Z$, whose energies shift with respect to $B = 0$ are given by

$$\Delta E = \mu_B (g_J m_J + g_I m_I) B / \hbar \quad (\text{A.10})$$

To take into account the hyperfine splitting, we calculate at lowest order

$$\delta E = \langle m_J, m_I | H_{\text{hf}} | m_J, m_I \rangle \quad (\text{A.11a})$$

$$= a_{\text{hf}} m_I m_J + b_{\text{hf}} \frac{9m_I^2 m_J^2 - 3I(I+1)m_J^2 - 3J(J+1)m_I^2 + IJ(I+1)(J+1)}{4IJ(2I+1)(2J+1)} \quad (\text{A.11b})$$

For Potassium, this regime is valid as soon as $B > 50 \text{ G}$ for $P_{3/2}$ states (resp. $B > 200 \text{ G}$ for $P_{1/2}$ states). For Lithium, this regime is valid as soon as $B > 3 \text{ G}$ for $P_{3/2}$ states (resp. $B > 30 \text{ G}$ for $P_{1/2}$ states). It is therefore the relevant expression to estimate the shifts at typical biases of the experiment (see A.1).

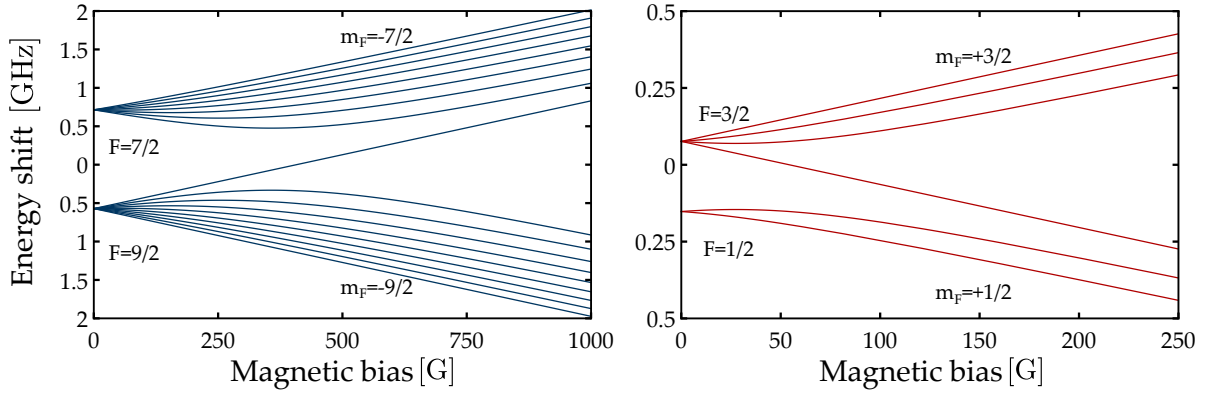


Figure A.2: Zeeman shift of the $S_{1/2}$ manifold of ^{40}K (left) and ^6Li (right) as a function of the magnetic field, as estimated by the Breit Rabi formula (A.15)

A.1.4 Breit-Rabi formula

For S orbitals, as the angular momentum $L = 0$, the Zeeman effect can be analytical calculated as show in [Breit and Rabi 1931]. Using standard notations, we rewrite

$$H = H_0 + a_{\text{hf}} \left(I_z J_z + \frac{1}{2} (I_+ J_- + I_- J_+) \right) / \hbar^2 + \mu_B B (g_J J_z + g_I I_z) / \hbar \quad (\text{A.12})$$

Since $J = 1/2$, we can expand the matrix expression of the Hamiltonian over the basis $\{|n, L = 0, S, J = S, I, m_J, m_I\rangle\} = \{|m_J, m_I\rangle\}$:

$$H - H_0 = \begin{pmatrix} E(m_F = -(I + \frac{1}{2})) & 0 & 0 & 0 & 0 \\ 0 & M_{11}(-I) & M_{12}(-I) & 0 & 0 \\ 0 & M_{21}(-I) & M_{22}(-I) & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & E(m_F = (I + \frac{1}{2})) \end{pmatrix} \quad (\text{A.13})$$

where the M matrices have the following form

$$M_{11}(m_I) = a_{\text{hf}} m_I / 2\hbar^2 + \mu_B B \left(\frac{g_I}{2} + m_I g_I \right) / \hbar, \quad (\text{A.14a})$$

$$M_{22}(m_I) = -a_{\text{hf}} (m_I + 1) / 2\hbar^2 + \mu_B B \left(-\frac{g_I}{2} + (m_I + 1) g_I \right) / \hbar, \quad (\text{A.14b})$$

$$M_{12}(m_I) = a_{\text{hf}} \sqrt{I(I+1) - (m_I+1)m_I} / 2\hbar^2 = M_{21}(m_I). \quad (\text{A.14c})$$

This block matrix can be easily diagonalized: the stretch states $m_F = \pm (I + 1/2)$ are already eigenstates and each subspace $\{|-1/2, m_I\rangle, |+1/2, m_I - 1\rangle\}$ with a constant m_F is stable. The

corresponding eigenvalues give the Zeeman shift:

$$\Delta E = \begin{cases} \frac{a_{\text{hf}}}{2\hbar^2} I \pm \frac{\mu_B}{\hbar} \left(\frac{g_I}{2} + g_I I \right) B & \text{for } m_F = \pm \left(I + \frac{1}{2} \right) \\ -\frac{a_{\text{hf}}}{4\hbar^2} + g_I \mu_B m_F B \pm \frac{a_{\text{hf}} \left(I + \frac{1}{2} \right)}{2} \sqrt{1 + \frac{2\mu_B (g_I - g_J) m_F B}{a_{\text{hf}} \left(I + \frac{1}{2} \right)^2} + \frac{\mu_B^2 (g_I - g_J)^2 B^2}{a_{\text{hf}}^2 \left(I + \frac{1}{2} \right)^2}} & \text{for } |m_F| < \left(I + \frac{1}{2} \right) \end{cases} \quad (\text{A.15})$$

Their values are plotted on figure [A.2](#).

A.1.5 Remark on notations

As shown in this appendix, the hyperfine states are eigenstates of the system only at zero field. The projected angular momentum m_F is therefore not a good quantum number generally speaking, when arbitrary magnetic field are applied. Nevertheless, we will refer to those numbers to label without ambiguity the eigenstates of the Zeeman Hamiltonian. Strictly speaking, the spin-state $|m_F\rangle$ should be understood as the eigenstate adiabatically connected to $|m_F\rangle$ at zero field.